

THERMODYNAMICS OF HYPERON STARS

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ABSTRACT

For a single-component perfect Fermi gas we used the numerical programme for the equation of state given by Bauer. For a star of hot non-degenerate neutron gas we calculated the deviations of the internal structure with regard to a totally degenerate neutron star. For a multi-component perfect gas with an exponential-type elementary particle spectrum we present the equation of state. The highest possible temperature is $T_0 = 2 \times 10^{12}$ K, where the total mass density diverges. For the central region of hyperon stars, in contrast to other authors, we can prove that the time component of the metric tensor has no singularity, and that the velocity of sound tends to zero (instead of rising above the velocity of light).

INTRODUCTION

We are studying the internal structure of hot neutron stars, which are in fact hyperon stars. Our main interest is directed towards the peculiar singularities in the centres of these stars. For this purpose we need an equation of state which can be used up to very high total energy densities ($\rho \gg 10^{14}$ g/cm³). The matter in such a state consists no longer of neutrons only, but also contains innumerable heavier particles and resonances. For in every elementary scattering process new particles can be produced if there is sufficient energy, and if the well-known conservation laws are not violated. Therefore detailed calculations of hot hyperon stars have to deal with the whole spectrum of elementary particles up to very high masses and their interactions. Hansen¹ has made a calculation, which is based on all particles with masses up to 1317 MeV (e, μ , π , n, p, Δ , Λ , Σ , Ξ), taking into account the conservation laws of the number of baryons, of electronic leptons, muonic leptons and of the total charge. By using a variational approach he proves that, as higher densities are approached ($\rho > 10^{17.5}$ g/cm³) the anti-particles as well as the leptons die out.

Our aim was to continue Hansen's calculation up to even higher densities, which may occur in the central region of a heavy neutron star. We claim that it is not possible to restrict the calculations to a definite number of different elementary particles, but that the possible production of any particle of the whole particle spectrum has to be included. For example, free heavy resonances normally disintegrate quickly, yet this behaviour is no longer observed in a dense region, if the degenerate Fermi distribution of the resulting particle is already fully occupied. In the intermediate region the disintegration is greatly impeded, depending on the chemical potentials and temperature, T .

Tsuruta and Cameron² calculated the static structure of a hyperon star with Hansen's equation of state for the degenerate case, especially for $T < 10^9 \text{K}$. Some of the results refer to densities up to 10^{21}g/cm^3 , where Hansen's definite mass spectrum is no longer applicable since the extrapolation of the known baryon spectrum should be assumed to be exponential^{3,4}. Therefore we have studied for an exponential baryon spectrum of this type the internal structure of a hyperon star up to infinite density and up to $T_0 = 2 \times 10^{12} \text{K}$, which then emerges as the lightest possible temperature.

For a conventional heavy neutron star of infinite central density the possible singularities of the metric tensor as well as of the velocity of sound have often been discussed. These peculiar features do not occur in hyperon stars.

EQUATION OF STATE OF A SINGLE GAS COMPONENT

For simplicity we assume that every component of the hyperon star can be treated as a perfect gas (i.e. we neglect the interactions). First we have to compose the equation of state of a single gas component. The particle number density n , the pressure P , and the kinetic energy density ε of a perfect gas are well-known:

$$\left. \begin{aligned} n &= \frac{8\pi}{h^3} \int_0^\infty p^2 dp [\exp(-\gamma + E/kT) + 1]^{-1} &= m^3 f_1 \left(\gamma, \frac{T}{m} \right) \\ \varepsilon &= \frac{8\pi}{h^3} \int_0^\infty p^2 dp [\exp(-\gamma + E/kT) + 1]^{-1} (E - mc^2) &= m^4 f_2 \left(\gamma, \frac{T}{m} \right) \\ P &= \frac{8\pi}{h^3} \int_0^\infty p^2 dp [\exp(-\gamma + E/kT) + 1]^{-1} p \frac{\partial E}{\partial p} &= m^4 f_3 \left(\gamma, \frac{T}{m} \right) \end{aligned} \right\} (1)$$

where the total energy E of a single fermion is related to its momentum p and its rest mass m by

$$E^2 = p^2 c^2 + (mc^2)^2 \tag{2}$$

and the parameter γ with the chemical potential μ by

$$\gamma := (\mu + mc^2)/kT \tag{3}$$

For our astrophysical application it is necessary to have for the integrals 1 a very fast computer programme of high accuracy and covering the whole area of the P/T plane. For some distinct regions series expansions have been given by Sommerfeld⁵, Chandrasekhar⁶, Guess⁷, Tooper⁸ and Bauer⁹. *Figure 1* shows the area where no series expansions are available. Three well-known limiting cases are of special interest. For extreme quantum degeneracy the integrals 1 can be solved analytically with the result that the equations of state $P(n, T)$ and $\varepsilon(n, T)$ respectively are independent of temperature. For the limiting cases of extreme non-relativistic degeneracy the following result is obtained:

$$\varepsilon = \frac{3}{2}P, \quad P \sim n^{\frac{5}{3}}, \quad P \sim \rho^{\frac{5}{3}} \tag{4}$$

THERMODYNAMICS OF HYPERON STARS

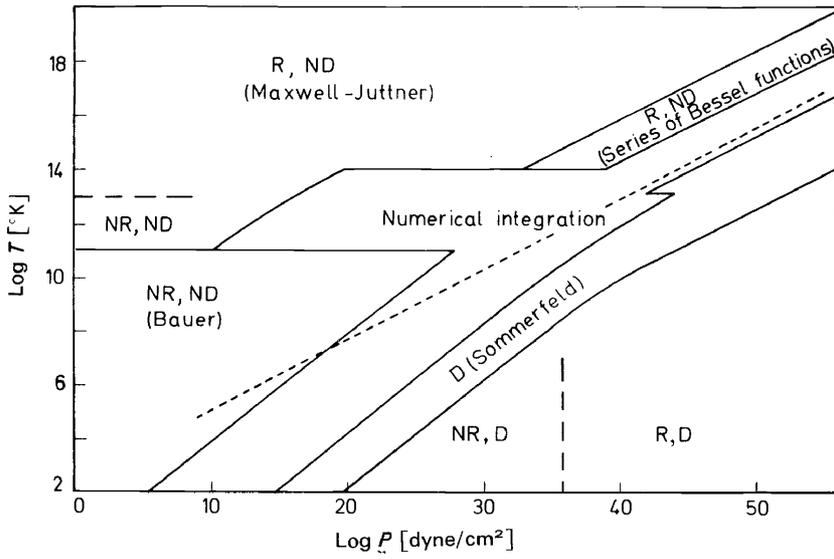


Figure 1. Areas of different ways of calculating the Fermi integrals for neutrons. Above the broken line the radiation pressure overwhelms that of a neutron gas. N non-, R-relativistic, D degenerate.

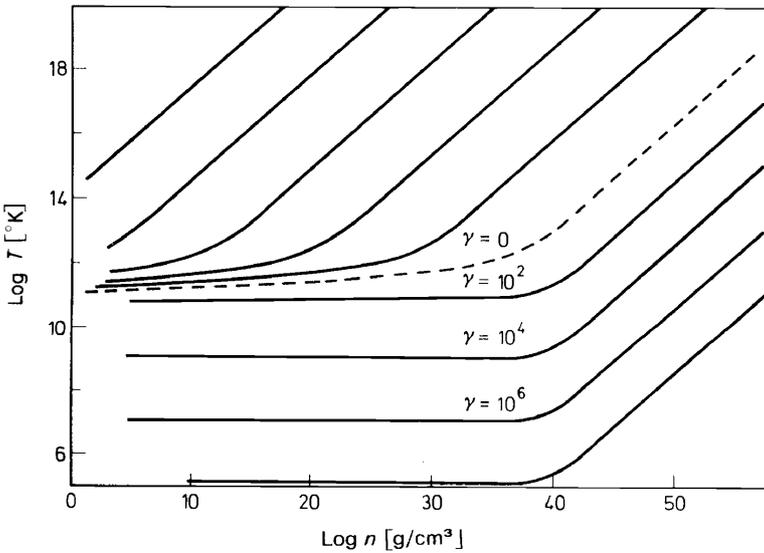


Figure 2. Lines of constant γ of a perfect neutron gas. They describe also the radial structure of the star, since the density declines monotonically and γ is constant throughout the star.

and for the extreme relativistic case:

$$\varepsilon = 3P, \quad P \sim n^{\frac{4}{3}}, \quad P \sim \frac{1}{3}\rho, \quad n \sim (T\gamma)^3 \quad (5)$$

and for non-relativistic non-degeneracy ($\mu/mc^2 \ll -1, mc^2/kT \ll 1$):

$$\varepsilon = \frac{3}{2}P, \quad P = nkT, \quad P = \frac{\rho}{m}kT, \quad n = \frac{2}{h^3}(2mkT)^{\frac{3}{2}} \exp\left(\gamma - \frac{mc^2}{kT}\right) \quad (6)$$

For matter in thermal equilibrium the parameter γ and $T\sqrt{g_{00}}$ (with g_{00} being the time component of the metric tensor in general relativity) are constant, as has been shown by Balazs¹⁰, Ehlers¹¹ and Ebert¹². For a neutron gas we have plotted in *Figure 2* the $T(n)$ curves as a function of γ . Since in a neutron star the density decreases with increasing distance from the centre, *Figure 2* shows that the temperature reaches a quasi-constant value in the outer parts, where $n \Rightarrow 0$.

HOT NEUTRON STARS

Oppenheimer and Volkoff¹³ discovered in 1939 that stars of a totally degenerated neutron gas are stable only if their total mass is less than a

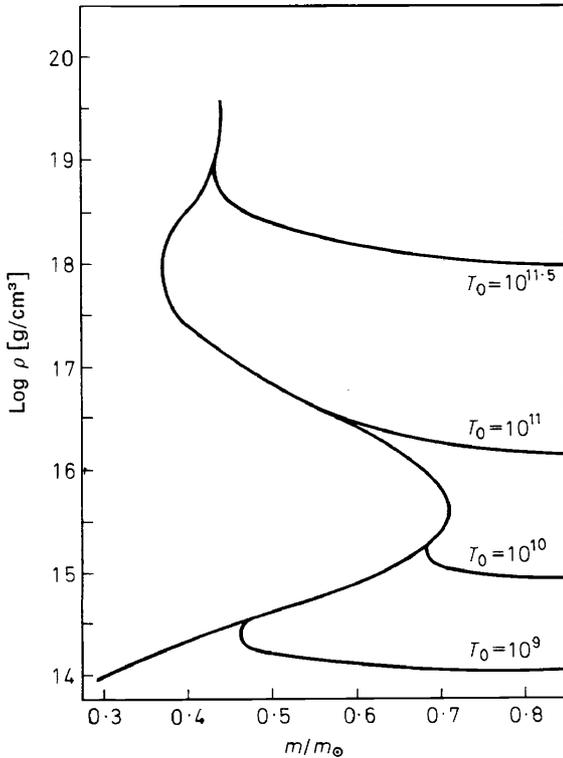


Figure 3. Derivations from the Oppenheimer-Volkoff curve (oscillating line) with increasing central temperature T_0 for a hot neutron star.

finite limiting mass m_G (see *Figure 3*). If $m > m_G$, a degenerate star will probably undergo a gravitational collapse, since in that case there is no static solution. But there are static solutions for partially degenerated neutron stars.

The structure of a radially symmetric static neutron star in thermal equilibrium is determined by the general relativistic field equations of Einstein. Using two formal parameters m^* and r they can be transformed into:

$$\frac{dm^*}{dr} = 4\pi r^2 \rho^*, \quad \frac{dT}{dr} = -T \frac{m^* + 4\pi r^3 P^*}{r(r - 2m^*)} \tag{8}$$

This system of differential equations can be solved uniquely if γ and the equations of state $\rho^* = \rho^*(\gamma, T)$ and $P^* = P^*(\gamma, T)$ are determined, and with the boundary values:

$$m^*(r = 0) = 0, \quad T(r = 0) = T_0 \tag{9}$$

In these equations the asterisk denotes that the quantity is measured in the natural units ($c = G = 1$).

m^* is chosen so that at a sufficient distance from the star the curved space becomes asymptotically flat and the motion of a sample is governed by Newton's laws. Then m^* and r can be identified by the mass and the radius respectively. Since $2m^* \ll r$ for static stars, for those stars in thermal equilibrium the temperature gradient is negative as shown in equations 8. This resembles the radial decrease of the gravitation potential which causes the temperature not to be constant in thermal equilibrium on account of the

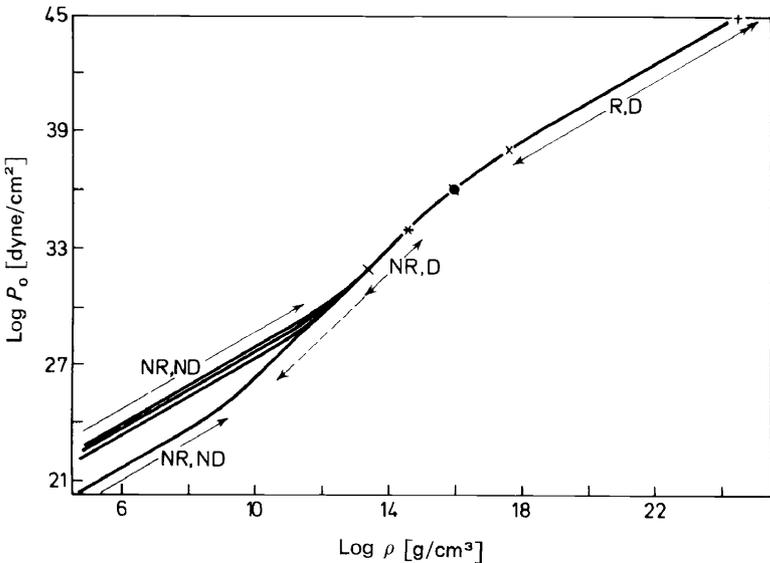


Figure 4. The equation of state of a neutron gas $P(\rho)$ for different fixed γ , or the relation of P and ρ in the radial direction of five stars with equal central temperature but different $\rho_0 = \rho(\gamma, T_0)$. Only the outer parts of the star depend on the degeneracy parameter γ .

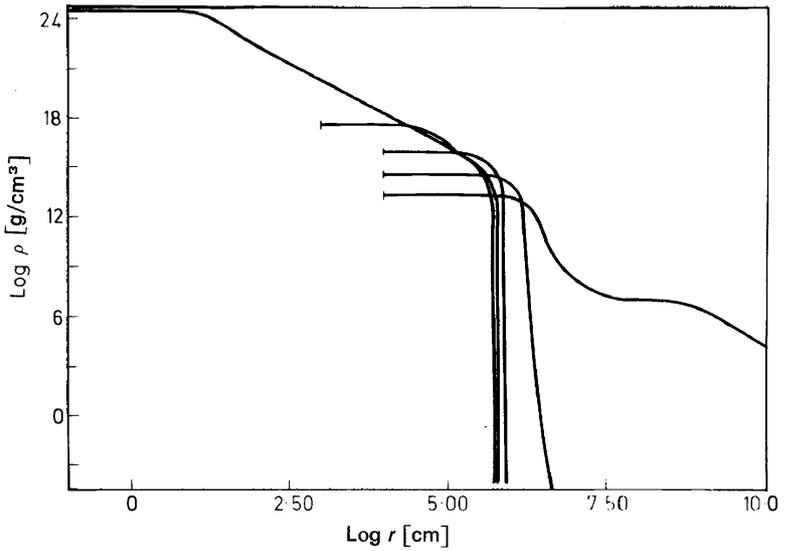


Figure 5. The density of the five selected stars as a function of r . The different behaviour in the outer parts of the star is caused by different degeneracy parameters γ . The total mass of the star becomes very great when the neutron gas in the high density region $\rho > 10^{12}$ g/cm³ is no longer degenerate.

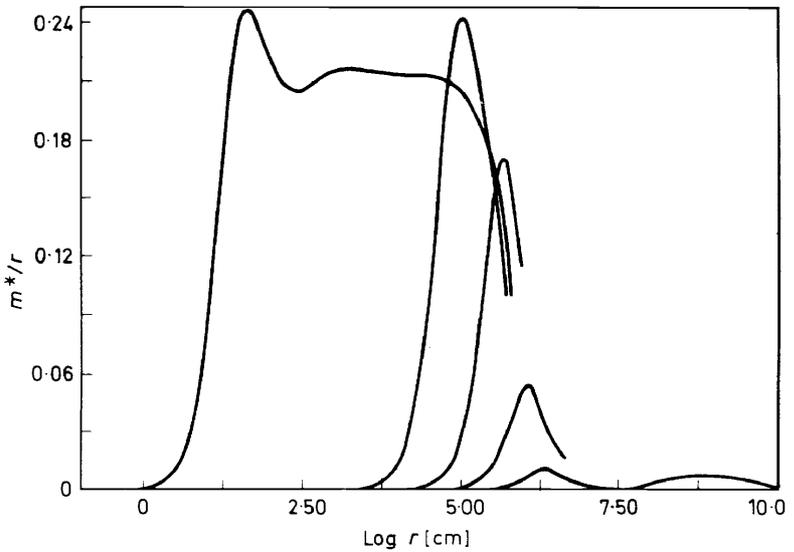


Figure 6. The parameter m^*/r as a function of the radius r for the five selected stars. In the classical theory the gravitational potential, and in Einstein's theory, the metric component g_{rr} , is related with m^*/r .

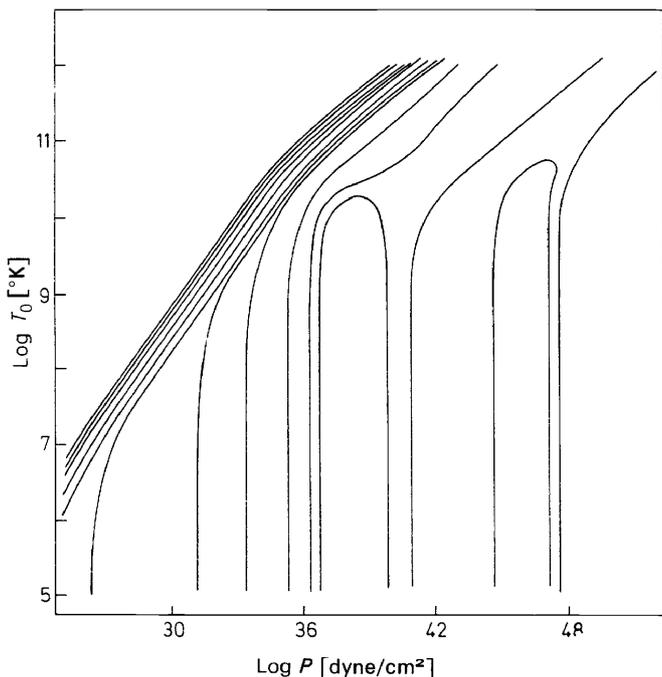


Figure 7. Contour lines of the finite radius R as a function of the central pressure P_0 and temperature T_0 . It is evident that in a great area R does not depend on the temperature inside the star. This was the reason for other authors to neglect the influence of the temperature

general relativistic red shift. As a consequence of the constancy of γ throughout the star, the star structure is determined by the respective γ -line in Figure 2, and by the differential equation 8 the parametrization of the radius is fixed. Our results are given in Figures 3 to 7. In Figure 3 the lines of equal central temperature are presented as a function of central density ρ_0 and relative total mass m/m_\odot (m_\odot being the mass of the sun). The great deviations of the Oppenheimer–Volkoff behaviour are registered only at high central temperatures, when a large part of stellar matter is no longer degenerated. This is illustrated in Figure 4. For high central densities the equation of state remains the same. With regard to the radial structure of the density the deviations due to the increasing non-degeneracy in the outer parts of the star are shown in Figure 5. Figure 6 makes the complexity of the peculiar internal structure of neutron stars quite evident. Figure 7 shows the contour lines of the radius r in the T_0/P_0 plot. Only for low central pressure and high temperatures does the structure of the star greatly depend on T_0 . This diagram is more suitable for discussing the influence of the non-degeneracy on the radius of the star than the usual vortex diagram. We would like to point out that the density decline is extremely steep in the immediate vicinity of the centre for stars with high ρ_0 and near the surface for degenerate stars.

MULTI-COMPONENT GAS

For the study of a hyperon star an equation of state for a multi-component perfect gas is needed.

The thermal equilibrium conditions $T = \text{const.}$ and $\gamma = \text{const.}$ for a multi-component gas system in a fixed volume can be derived by maximization of the entropy $S = \sum_{i=1} S_i$ of the whole system.

$$\delta S_i = \frac{\delta E_i}{KT_i} - \frac{\mu_i + m_i c^2}{KT_i} \delta N_i \tag{10}$$

If the total energy $E = \sum_i E_i$ and the total particle number $N = \sum_i N_i$ are conserved, and δE_i and δN_i can be varied independently, the above-mentioned equilibrium conditions are obtained. The equation of state of the multi-component system can be calculated directly from those of the single gas components, if γ and T are known.

In order to apply this model to the matter of hyperon stars we have to assume that the interaction between the particles can be neglected, and that the total number of particles is conserved in every elementary process. The strong interaction between some neighbouring particles can be taken into account by defining these as one new particle in terms of thermodynamics ('molecule'). Moreover, we need the abundance distribution $a(m)$ of the components within matter at high densities. It has been confirmed by Hansen's¹ calculation that the anti-particles and the Bosons diminish in relation to the increase of the Fermions, to which we have restricted our calculation.

The abundance distribution of the Fermions $a(m)$ is defined by the product

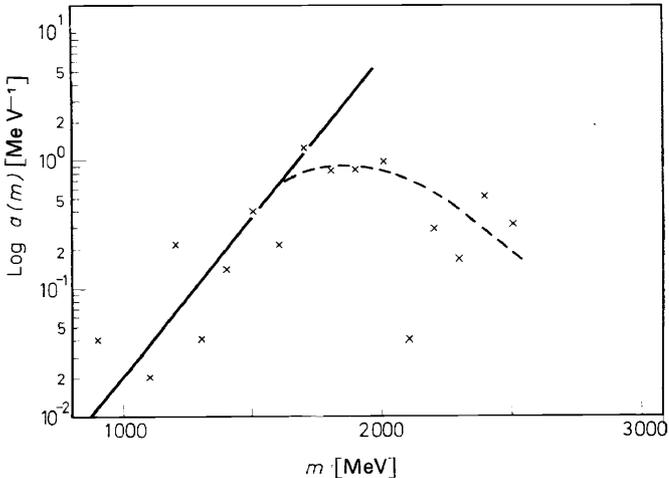


Figure 8. The abundance distribution $a(m)$, the product of the number particles per mass interval and the factor of multiplicity $(1 + 2i)(1 + 2j)$, is plotted against the baryon mass. The points are results of the statistics of those particles which are known, the dashed line is the interpolation line of the discrete experimental abundance distribution. The straight line is the theoretical continuous abundance distribution, suggested here.

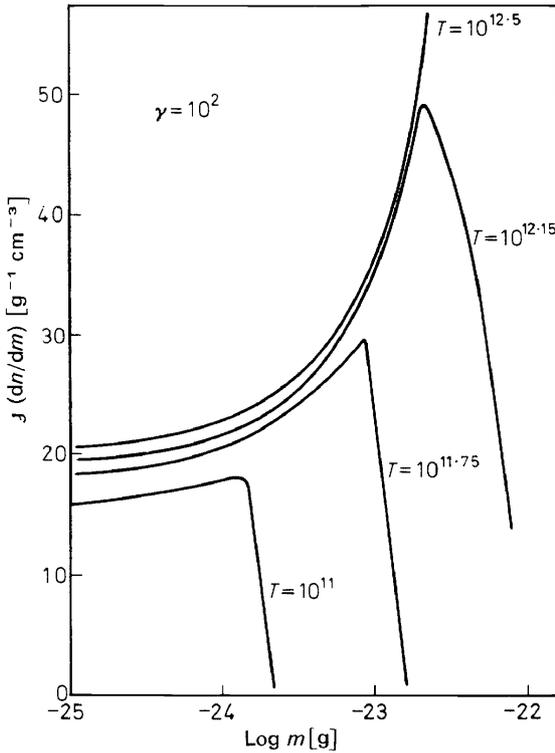


Figure 9. The particle number density per mass interval dn/dm [g^{-1}] is plotted against m . While γ for all lines is fixed, the temperature varies from one line to another. Thus this series of distribution functions describes the deviations of this function if an observer moves towards the centre of the star.

of the number of baryon components per mass unit and their multiplicity $(1 + 2i) \times (1 + 2j)$ with i and j being the spin and the isospin respectively. We confine ourselves to baryons and their heavier resonances. Their abundance distribution has been measured up to about 3 GeV. In Figure 8 interpolation of the experimental data¹⁶ resembles very much the exponential behaviour in the area where it is likely that all baryons are detected. If it is assumed that this exponential behaviour holds for all masses—this has not been contradicted up to now by experiments, and several theoretical arguments^{3,4} have been put forward in favour of it—it holds also if only strongly interacting particles are counted as new ones thermodynamically.

For the quantitative fit of $a(m)$ between 1 and 2 GeV we used not only particles with well-known (i, j, m) but also those where some of these quantities were lacking and were to be interpolated in a simple fashion with the result:

$$a(m) = \exp\left(\frac{mc^2}{kT_0} + 52\right), \quad T_0 = 2.0 \times 10^{12} \text{ } ^\circ\text{K} \quad (11)$$

This parameter T_0 is approximately the same as the 'highest possible temperature in nature' of Hagedorn³. After substituting the discrete abundance distribution by $a(m)$, the total number density

$$n(\gamma, T) = \int_{m_n}^{\infty} dm \exp\left(\frac{mc^2}{kT_0} + 52\right) \frac{1}{2} m^3 f_1\left(\gamma, \frac{T}{m}\right) \quad (12)$$

together with an analogue formula for the total pressure P and for the energy density ε yields approximately the equation of state of the multi-component gas. In Figure 9 $\log (dn/dm)$ is plotted versus $\log (m)$ for $\gamma = 10^2$ and different T . The maximum of any curve exhibits the most frequent particle component at that temperature. For $T \Rightarrow T_0$ the matter curdles, i.e. the total rest mass energy grows faster than the kinetic energy. The components of the heaviest particles for a given γ and T are not degenerated. Therefore, with the asymptotic expression 7

$$\frac{dn}{dm} = \frac{1}{h^3} (2\pi mkT)^{\frac{3}{2}} \exp\left\{\gamma + \frac{mc^2}{kT_0} \left(1 - \frac{T_0}{T}\right) + 52\right\} \quad (13)$$

for heavy masses, the total particle number density $n(\gamma T)$ diverges for $T \Rightarrow T_0$ (see Figure 11). In Figures 10 and 11 the equation of state for fixed γ of a

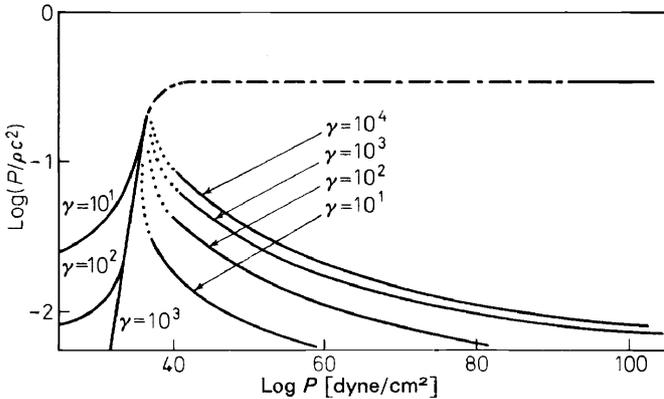


Figure 10. Differences between the perfect neutron gas (chain-dotted) and the hyperon gas (full line). For increasing pressure, P/ρ tends to the value one third for the former, in the latter case to zero. The equation of state at high pressures depends on the degeneracy parameter γ .

multi-component gas is compared with that of a neutron gas. While P/ρ for all γ s in the case of the neutron gas approaches asymptotically the value one third with increasing total pressure, P/ρ declines in the case of the hyperon gas. The exponential factor z , defined by

$$P \sim \rho^z \quad (14)$$

in this case is $\ll 1$ and the equation of state depends in the high pressure region on the degeneracy parameter (the heaviest gas components are not degenerated). Although we did not calculate transition states between the

multi-component gas (with the continuous abundance distribution accepted here) and the neutron gas, the qualitative behaviour as plotted in *Figures 10 and 11* (dotted lines) seems to be evident. Calculations of multi-component

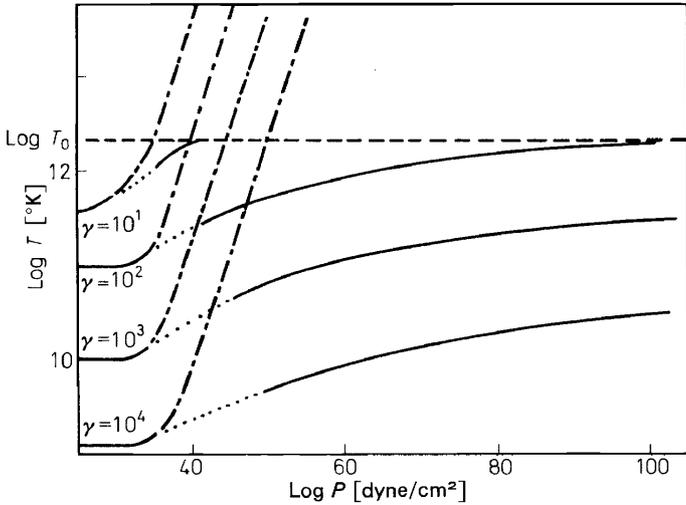


Figure 11. Pressure P dependence of the temperature of a neutron gas (chain-dotted) and a hyperon gas (full line). The derivations are important at low temperatures too.

gas systems have been made by Hagedorn³ too, but in order to keep the calculations analytical, he restricts them to the case $\gamma = 0$. The results may therefore be applicable to the big bang³, but not to the structure of the stars.

CENTRAL SINGULARITY OF NEUTRON AND HYPERON STARS

For neutron stars of infinite central density it is known for all equations of state applied up to now that g_{00} diverges to zero for $r = 0$ with increasing central pressure.

Using the energy-momentum tensor of a static ideal fluid $T_{00} = \rho$, $T_{ij} = -P\delta_{ij}$ and the general line element of a static radially-symmetric star

$$ds^2 = g_{00} dt^2 - g_{rr} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{15}$$

Einstein field equations yield the equation

$$g_{00/r=0} = \left(1 - \frac{2M^*}{R}\right)^{\frac{1}{2}} \exp\left(-\int_0^{P_0} \frac{dP}{P + \rho c^2}\right) \tag{16}$$

integrated in the radial direction. The factor $(1 - 2M^*/R)^{\frac{1}{2}}$ is chosen so that g_{00} is continuous in the radial direction on the surface of the star ($P_{/r=R} = 0$). So the integral diverges in its upper limit, if the exponential factor z of equation 14 is greater than one. Then $g_{00/r=0} = 0$ for $P_0 \Rightarrow \infty$. This zero

of the temporal component of the metric tensor in the centre of the star, a peculiar effect ('singularity') of general relativity, may be called 'zerolarity'. This 'zerolarity' will not occur if the exponential factor z is less one, since then the integral converges and is finite.

Ehlers¹¹ has proved (in a general relativistic kinetic gas theory) that the equilibrium conditions $\gamma = \text{const.}$ and $T\sqrt{g_{00}} = \text{const.}$ are valid for multi-component systems too. But since in our open multi-component gas the temperature cannot rise above T_0 (even if the energy density should rise to infinity) we are sure that in thermal equilibrium g_{00} must have a positive minimum at the centre of the star and indeed the integral 14 converges for the equation of state of our system.

Other authors¹⁵ have obtained the strange result that for very high densities the velocity of sound $v_s = (dP/d\rho)^{\frac{1}{2}}$ seems to surpass the velocity of light. This result has been arrived at by taking the repulsive part of the nuclear forces into account. Then the pressure rises more quickly than the total mass energy ρ . In our multi-component system such contradiction does not occur. Moreover, the sound velocity decreases with increasing density.

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